Deriving the Laplacian in Spherical Coordinates

Mert Okyay

Introduction

Here we shall compute the Laplacian Δ in spherical coordinates, using the crowd favourite index notation.

1 Derivation

The (coordinate-free) Laplacian in abstract index notation in terms of the metric and partial derivatives $\frac{\partial}{\partial x^i}$ is

$$\Delta = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^a}\left(\sqrt{g}\; g^{ab}\frac{\partial}{\partial x^b}\right) \tag{1}$$

1. We will first construct the metric by $g_{ij} = \hat{e}_i \cdot \hat{e}_j$ in the spherical basis.
2. We will then compute $\sqrt{g}$ which is shorthand for $\sqrt{\det(g^{ab})}$.
3. We will then simplify the expression to look like the usual laplacian.

1.1 Constructing the g^{ab}

The representation of the usual position vector in Cartesian coordinates is

$$V = V^{\mu}\hat{e}_{\mu} = V^x\hat{e}_x + V^y\hat{e}_y + V^z\hat{e}_z \tag{2}$$

which to convert to Spherical coordinates (r, θ, ϕ) we need

$$\begin{aligned} x &= r\cos\phi\sin\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\theta \end{aligned}$$

which gives (2) in the new basis:

$$V = r\cos\phi\sin\theta\hat{e}_x + r\sin\phi\sin\theta\hat{e}_y + r\cos\theta\hat{e}_z \tag{3}$$

which we can use to derive the orthogonal basis by the use of the formula

$$\frac{\partial}{\partial x^i}V = \hat{e}_i \tag{4}$$

$$\begin{aligned} \hat{e}_r &= \cos\phi\sin\theta\hat{e}_x + \sin\phi\sin\theta\hat{e}_y + \cos\theta\hat{e}_z \\ \hat{e}_\phi &= -r\sin\phi\sin\theta\hat{e}_x + r\cos\phi\sin\theta\hat{e}_y + 0\hat{e}_z \\ \hat{e}_\theta &= r\cos\phi\cos\theta\hat{e}_x + r\sin\phi\cos\theta\hat{e}_y - r\sin\theta\hat{e}_z \end{aligned}$$

Then we construct the $g_{ij} = \hat{e}_i \cdot \hat{e}_j$ where the orthogonality of the vectors makes off-diagonal elements vanish:

$$\hat{e}_r \cdot \hat{e}_r = \cos^2\phi\sin^2\theta + \sin^2\phi\sin^2\theta + \cos^2\theta = 1 \tag{5}$$

$$\hat{e}_\phi \cdot \hat{e}_\phi = r^2\sin^2\phi\sin^2\theta + r^2\cos^2\phi\sin^2\theta = r^2\sin^2\theta \tag{6}$$

$$\hat{e}_\theta \cdot \hat{e}_\theta = r^2\cos^2\phi\cos^2\theta + r^2\sin^2\phi\cos^2\theta + r^2\sin^2\theta = r^2 \tag{7}$$

giving

$$g_{ab} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 \sin^2\theta & 0 \\ 0 & 0 & r^2 \end{bmatrix} \tag{8}$$

Now we need to "raise" the indices of g to obtain the contravariant metric tensor, obeying the rule $g^{ij} g_{ij} = \delta^i_j$ The obvious such matrix is

$$g^{ab} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2 \sin^2\theta} & 0 \\ 0 & 0 & \frac{1}{r^2} \end{bmatrix} \tag{9}$$

1.2 Computing the $\sqrt{g}$

The final ingredient is $\sqrt{g}$ which is shorthand for $\sqrt{\det(g_{ab})}$, which is simply

$$\sqrt{g} = \sqrt{r^4 \sin^2\theta} = r^2 \sin\theta \tag{10}$$

1.3 Grand Finale

Using the formula in (1) we bring it home,

$$\Delta = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \phi}\left(r^2 \sin\theta \frac{1}{r^2 \sin^2\theta} \frac{\partial}{\partial \phi}\right) + \frac{\partial}{\partial \theta}\left(r^2 \sin\theta \frac{1}{r^2} \frac{\partial}{\partial \theta}\right)\Bigg) \tag{11}$$

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial^2 \phi} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) \tag{12}$$

is the Laplace operator in spherical coordinate basis.